Spectral Density Shaping of Quantisation Error Using Dithering

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EUSIPCO 2024
32nd EUROPEAN SIGNAL PROCESSING CONFERENCE
AUGUST 26-30, 2024 • LYON, FRANCE

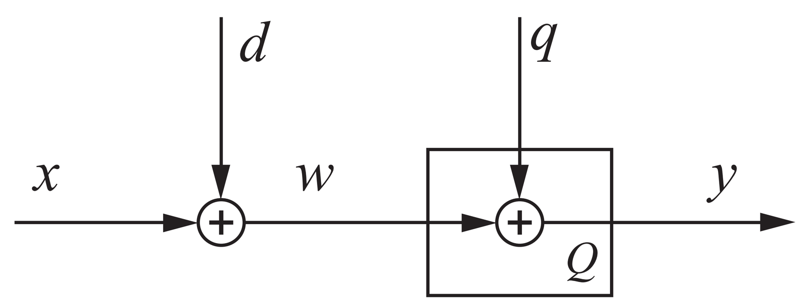
• Background:

Quantisation and **re-quantisation** are fundamental operations in digital signal processing, digital-analogue conversion, power electronics and measurement systems. **Error** is introduced since **only a discrete subset of values can be represented**. Several methods can be used **to shape** the power spectral density **(PSD)** of the error due to quantisation. Such methods include **ΔΣ-modulation**, model predictive control (**MPC**) and Learning control (**LC**). **Shaping** the error PSD **can be used to reduce the effective power content** of error due to quantisation **by additional filtering** in the frequency domain **where the error power is concentrated**.

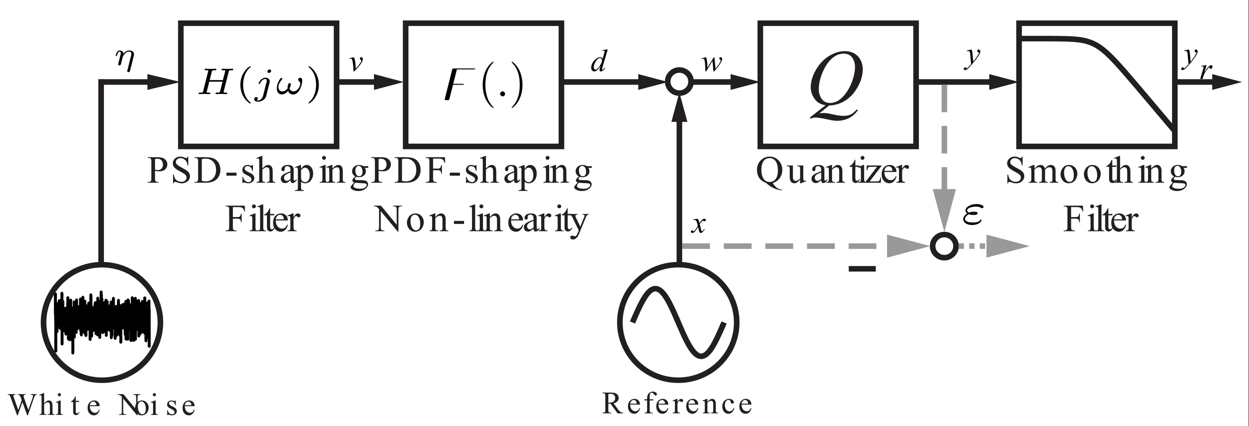
• Method:

Dither is a high-frequency periodic or stochastic signal introduced into a system to modify its non-linear characteristics.

- Quantisation error: $q(w) \triangleq Q(w) - w = y - w$



The method relies on the use of a **linear noise colouring filter** $H(j\omega)$ and **a static non-linear transform** $F(\cdot)$ to generate a dither signal with **a joint specification** of the probability density function (**PDF**) and the **PSD**.



- Total error due to quantisation : $\varepsilon \triangleq y - x$

• Contribution:

Results from **[1]** and the method from **[2]** are used to provide a **novel solution** to the problem of **shaping the PSD of the total error** due to uniform quantisation when applying a non-subtractive dither signal. The paper demonstrates that making the error power **independent of the input** enables shaping the error PSD as a **separate problem.** Moreover, an **improved** performance **compared to existing** non-subtractive **dithering methods** is shown attainable for a **wide range** of PSDs.

• Design Objectives:

Generating a dither signal ***d*** which **both** linearizes the uniform quantiser ($\boldsymbol{E[y]} = \boldsymbol{x}$) **and** induces a desired PSD ***S(ω)*** for ***ε***.

• Existing Methods:

The best spectral shaping performance known to be achievable using non-subtractive dithering is found in **[1]**:

$$S_\varepsilon(\omega) = S_d(\omega) + \frac{\Delta^2}{12}\frac{2}{f_s} \tag{1}$$

It is limited by a quantisation noise floor $\frac{\Delta^2}{12}$.
Where $\boldsymbol{\Delta}$ is the least significant bit (**LSB**).

• Proposed Approach:

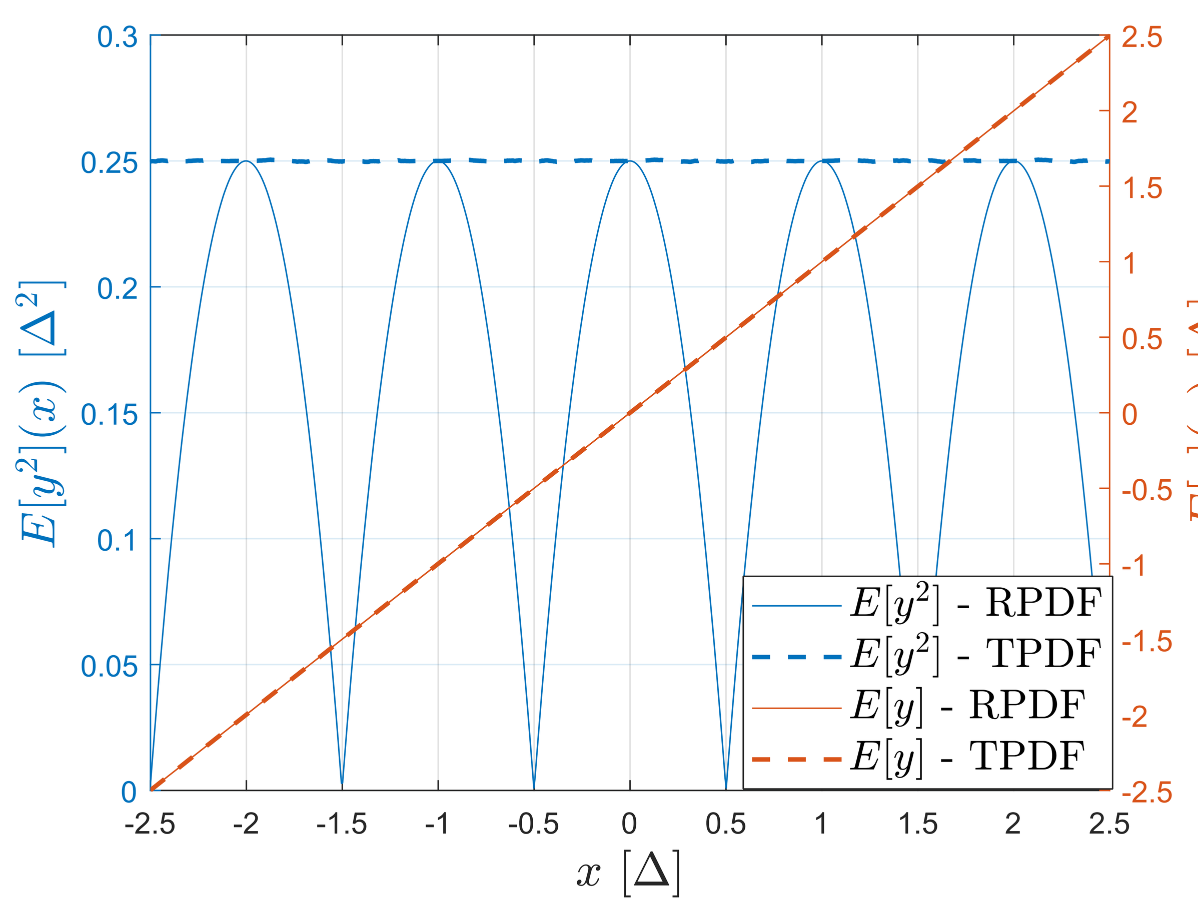
The method in **[2]** is adapted to work out the connection **between** the autocovariance functions (**ACFs**) $\phi(\tau)$ <u>of *υ*</u> and $s(\tau)$ <u>of *ε* *such that*</u> **the spectral smearing** due to the PDF-shaping non-linearity *F(·)* and the uniform quantiser *Q(·)* **is anticipated**. From [2], the autocorrelation functions (normalized ACFs) ***R(τ)*** <u>of *ε*</u> and ***ρ(τ)*** <u>of *υ*</u> are related as follows:

$$R = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{g(m,n,\rho)}{E[\varepsilon^2]} \tag{2}$$

$$(Q(F(m)+x)-x)(Q(F(n)+x)-x)dmdn.$$

- The choice of dither PDF (i.e. *F(·)*):

Rectangular PDF vs **Triangular** PDF



From **[1]**, we know that **both** RPDF and TPDF dither can linearise the uniform quantiser (since both have $E[\varepsilon](x) = 0$; i.e. $\boldsymbol{E[y]} = \boldsymbol{x}$). However, considering $s(\tau) = E[\varepsilon^2]R(\tau)$, the dither PDF choice should aim to **avoid** the **noise modulation** effect due to the **dependence** of the error power ($E[\varepsilon^2](x)$) over **x**. From the Fig. above, **TPDF** is the appropriate choice. Hence, solving (2) yields:

$$\rho(\tau) = \sin\left(\frac{\pi}{2}R(\tau)\right) \tag{3}$$

• Synthesis Limitations:

As long as the specified $S(\omega)$ corresponds to a $\Phi(\omega) \geq 0$ for all ω (PSDs must be non-negative), the desired shaping at the output stage is attainable. Unfortunately, $\Phi(\omega)$ is not guaranteed to be non-negative at all frequencies for every specified $S(\omega)$.

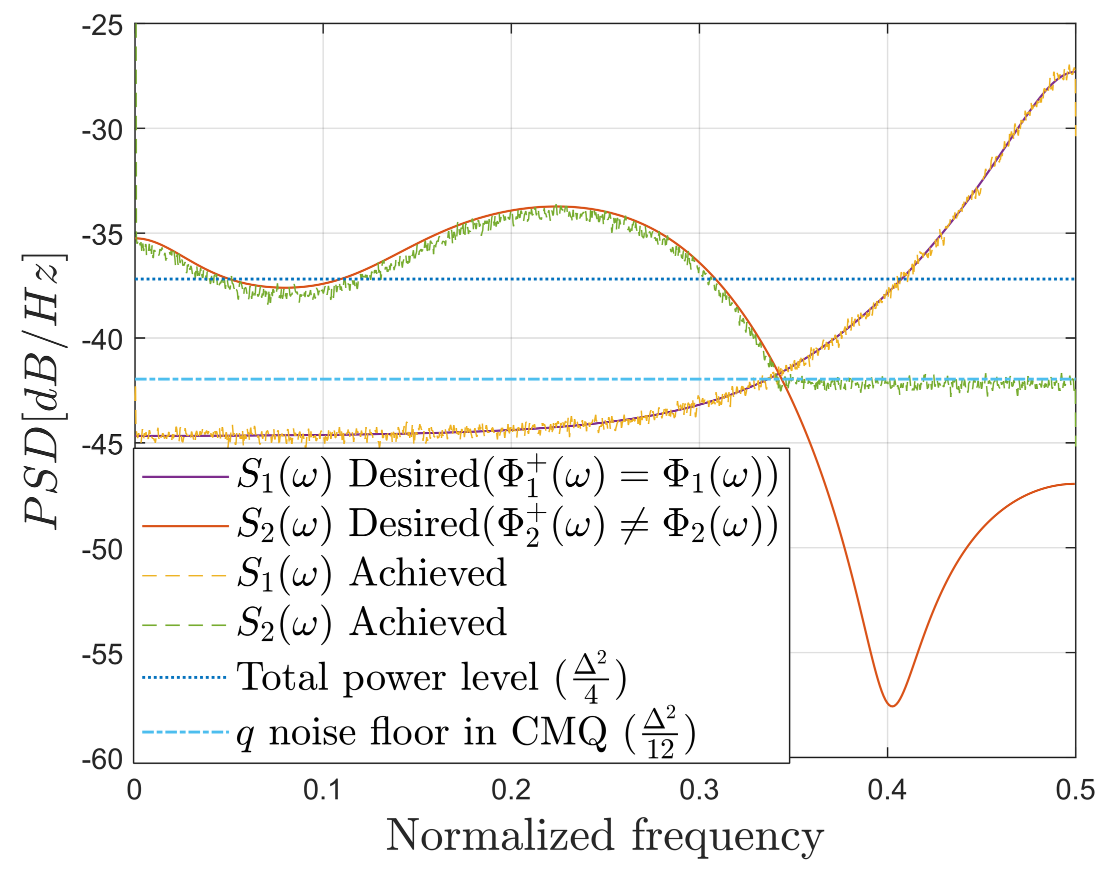
- Approximating the colouring filter $H(j\omega)$:

Let $\Phi^+(\omega) \triangleq \begin{cases} \Phi(\omega), & \Phi(\omega) \geq 0 \\ 0, & \Phi(\omega) < 0 \end{cases}$, (4)

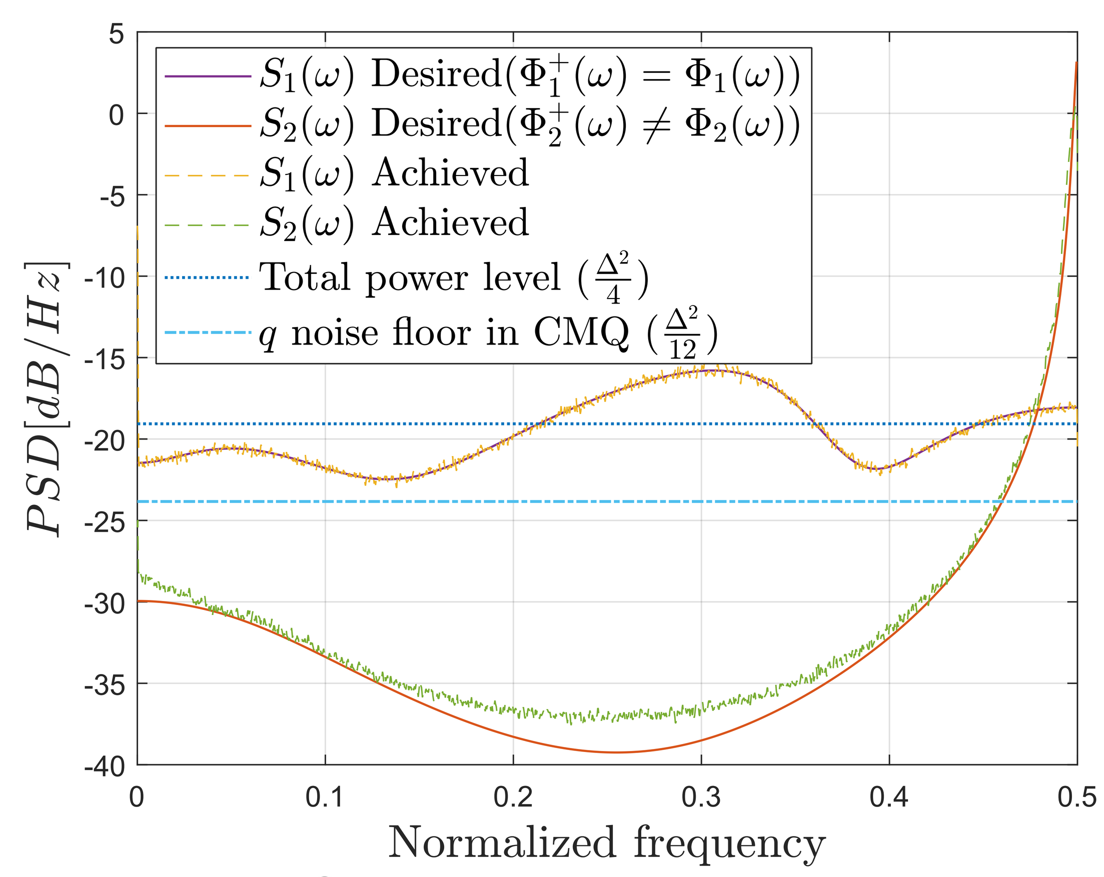
then: $H(j\omega) = \sqrt{\Phi^+(\omega)}$ (5)

• Results: $S(\omega) < \frac{\Delta^2}{12}$ **attainable!**

- 10 Bits uniform quantiser, x = x0.



- 7 Bits uniform quantiser, x = x1 ≠ x0



• References:

[1] R. A. Wannamaker, S. Lipshitz, J. Vanderkooy, and J. N. Wright, "A Theory of Nonsubtractive Dither," *IEEE Trans. Signal Process.*, vol. 48, no. 2, pp. 499–516, 2000.

[2] M. M. Sondhi, "Random processes with specified spectral density and first-order probability density," *Bell System Technical Journal*, vol. 62, no. 3, pp. 679–701, 1983.

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